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Fast, Accurate Frequency Estimators

The problem of estimating the frequency of a tone contaminated with noise appears in communications, audio, medical, instrumentation, and other applications. The fundamental tool for resolving this problem is the discrete Fourier transform (DFT) or its efficient cousin, the fast Fourier transform (FFT). When using either of them, a well-known tradeoff exists between the amount of time needed to collect data, the number of data points collected, the type of time-domain window used, and the resolution that can be achieved in the frequency domain. This article presents computationally simple algorithms that provide substantial refinement of the frequency estimation of tones based on DFT samples without the need for increasing the DFT size.

THE IDEA

When estimating the frequency of a tone, the idea is to estimate the frequency of the spectral peak k_{peak} (shown in Figure 1) based on three DFT samples: X_{k-1}, X_k, and X_{k+1}. If we selected k_{peak} by making it equal to the k index of the largest DFT magnitude sample, then the maximum estimation error in k_{peak} would be equal to half the width of the DFT bin. However, if we used the frequency-domain peak sample X_k, and one or two adjacent samples, the estimate of the peak location could be more accurate if we used simple best- or approximate-fit algorithms. In this article, we discuss solutions that provide a fractional correction term δ to be added to the integer peak index k to determine a fine estimate of the spectral peak location k_{peak} located at the cyclic frequency f_{tone}

$$k_{peak} = k + \delta \quad \text{and}$$
$$f_{tone} = k_{peak} f_s / N, \qquad (1)$$

where f_s is the time data sample rate in Hz and N is the DFT size. Note that δ can be positive or negative, and k_{peak} need not be an integer. This δ-corrected refinement of the original bin location estimate can be surprisingly accurate even in low signal-to-noise ratio (SNR) conditions.

SPECTRAL PEAK LOCATION ESTIMATION

Many spectral peak location estimation solutions with different computational complexities have been described [1]–[7]. Here we focus on three lesser known, but accurate and computationally simple, estimators. An example of a computationally simple peak location estimation that uses three DFT magnitude samples [8], [9] makes use of a correction term given by

$$\delta = \frac{(|X_{k+1}| - |X_{k-1}|)}{(4|X_k| - 2|X_{k-1}| - 2|X_{k+1}|)}. \qquad (2)$$

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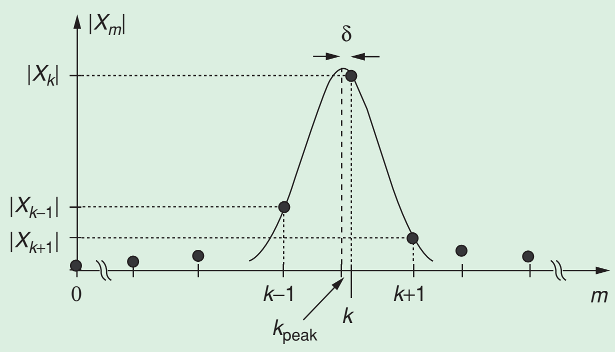
The expression is simple, but it is statistically biased and performs poorly in the presence of noise. Some simple changes to (2) improve its accuracy dramatically [4], for instance by using the complex DFT values rather than the magnitudes as follows

$$\delta = -\text{Re}\left[\frac{(X_{k+1} - X_{k-1})}{(2X_k - X_{k-1} - X_{k+1})}\right]. \qquad (3)$$

The accuracy of the spectral peak location estimation has been improved and the statistical bias of (2) has been eliminated. Even more, (3) provides potential for computation reduction by avoiding the nontrivial magnitude calculations in (2).



[FIG1] DFT magnitude samples of a spectral tone.

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[TABLE 1] CORRECTION SCALING VALUES FOR COMMON WINDOW TYPES USED WITH (4) AND (5).

WINDOW	*P*	*Q*
HAMMING	1.22	0.60
HANNING	1.36	0.55
BLACKMAN	1.75	0.55
BLACKMAN-HARRIS (THREE-TERM)	1.72	0.56

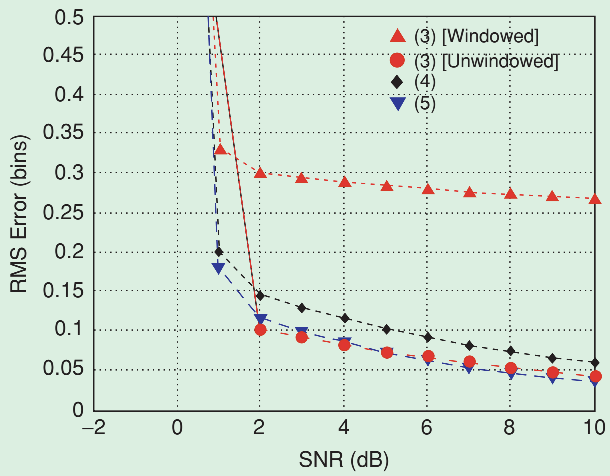
While the solution in (3) works well when a rectangular time-domain window is applied to the DFT's input samples, it is often beneficial or necessary to use nonrectangular windowing. One computationally simple alternative that is useful when time-domain data windowing has been used is given by

$$\delta = \frac{P(|X_{k+1}| - |X_{k-1}|)}{(|X_k| + |X_{k-1}| + |X_{k+1}|)}, \quad (4)$$

where the scaling constant P can be adjusted for different window functions [10]. However, (4) requires the computation of the DFT magnitude samples. Inspired by (3) and (4), a solution for use with nonrectangular windowed time samples has been suggested that does not require DFT magnitude computations; it is given by

$$\delta = \mathrm{Re}\left[\frac{Q(X_{k-1} - X_{k+1})}{(2X_k + X_{k-1} + X_{k+1})}\right], \quad (5)$$

where Q is a window-specific scaling constant [11]. Examples of scaling factors for (4) and (5) and common windowing functions are included in Table 1.

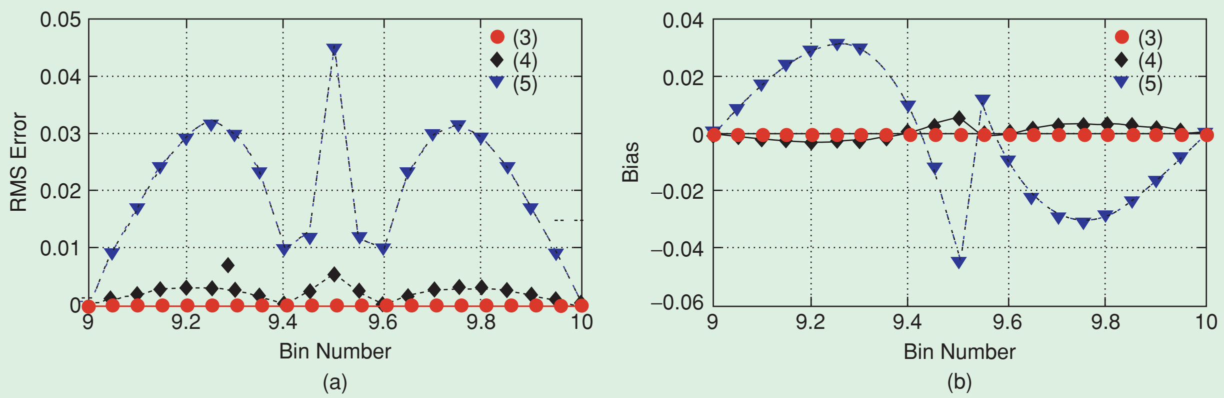


[FIG2] Algorithm RMS error performances versus tone SNR.

PERFORMANCE EVALUATION

The performance of the solutions (3)–(5) for a particular application can be expressed in terms of accuracy [root mean square (RMS) error], sensitivity to bias, and computational complexity, and depends on the type of windowing used. Estimators (4) and (5) provide less accuracy and have more highly biased outputs than (3), but their accuracy is better than (3) when applied to DFT samples from nonrectangular-windowed data. Figure 2 shows estimator RMS error for (3), (4), and (5) as a function of the tone's SNR when a tone at a 9.5-bin location is contaminated with white Gaussian noise. The performance of (3) is shown using unwindowed FFT input data and Hanning windowed FFT input data. All other algorithms use a Hanning window. Note the significant performance degradation of (3) when the data is windowed.

Figure 3 shows the simulation results of a tone whose spectral peak location was varied from bin 9 to bin 10 at small increments for a length-64 signal. The figure shows the RMS error and bias as functions of tone peak location. It can be seen that the performance of each algorithm changes depending on the tone offset within the bin. Table 2 includes a comparison of the computational complexities of the estimators (3), (4), and (5). All of them use additions or subtractions, but these are typically trivial to implement in hardware or software and are not considered significant for computational complexity comparison (as opposed to



[FIG3] Interbin performance in the absence of noise: (a) RMS error and (b) bias.

the number of multiplications, divisions, and magnitude computations).

CONCLUSIONS

Interbin tone peak location for isolated peaks can be performed effectively in the presence or absence of noise and with rectangular or nonrectangular windowing. The most appropriate solution for a particular application depends on the accuracy (RMS error), sensitivity to bias, the type of windowing used, and computational complexity. Among the estimators discussed, (3) is very accurate and provides a good complexity/ performance tradeoff as it requires only a single divide. Equations (4) and (5) are appropriate for use when the DFT input data has been windowed, although they do suffer from bias errors. Fortunately, the bias is predictable and it can often be removed with an additional operation.

[TABLE 2] COMPARISON OF THE ESTIMATORS (3), (4), AND (5).

EQ.	MULT.	DIV.	MAG.	RMSE	BIAS	REMARKS
(3)	4	1	0	MEDIUM, INCREASING WITH BIN OFFSET IN NOISE	NO	VERY GOOD BIAS PERFORMANCE
(4)	1	1	3	HIGH	YES	GOOD FOR WINDOWED APPLICATIONS
(5)	5	1	0	HIGH	YES	GOOD FOR WINDOWED APPLICATIONS

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