A Brief Examination of Current and a Proposed Fine Frequency Estimator Using Three DFT Samples

Eric Jacobsen
Anchor Hill Communications
June, 2015

Introduction and History

The practice of fine frequency estimation of a single tone using the three samples around the DFT output maximizer has progressed steadily over the preceding decades. A number of authors built on the early work of Kay, Rife and Boorstyn, et al. In the middle 1990s Quinn's estimators were considered superior, but were computationally expensive for real-time processors of the day. MacLeod and Jacobsen subsequently contributed estimators at about the same time in the late 1990s. Jacobsen's offered performance similar to Quinn's with reduced complexity and a resemblance to the quadratic interpolator arrangement. Jacobsen's estimator [1] was developed heuristically and presented without derivation.

Much later, in 2011, Candan [2] published a complete derivation of Jacobsen's estimator using infinite series, and added a bias-correction term based on the derived series expressions. This offered a significant reduction in the estimator bias that dominated the error performance at high SNR for low N. Low SNR performance was not affected. In 2013 Candan further improved on the bias performance using representations of the Fourier Coefficients [3]. The result was a statistically efficient estimator with no bias at high SNR.

In 2014 Liao and Lo [4] published an analysis using straightforward expressions for the DFT coefficients to apply a bias-correction term to a class of estimators that they referred to as QMJ, for Quinn, MacLeod and Jacobsen. An additional correction term to further improve Candan's first [2] estimator was also developed.

In all of the above developments a trend was apparent that there existed a general tradeoff between complexity and bias reduction. Jacobsen's estimator offered the minimum complexity but with some remaining bias. Candan's first estimator [2] reduced that bias with a simple correction term, and further reduced it in [3] at the cost of additional complexity. Liao's bias corrections provided improved performance over [1][2] but with more complexity than Candan's bias-free estimator in [3]. In general, it was evident that the removal of the residual bias cost additional complexity compared to Jacobsen's. This provided an interesting implementation tradeoff where the complexity could be included or removed depending on the SNR range of the application and the desired bias performance.

In 2015 Cedron proposed a new estimator [5], derived for real-valued tones, which is currently being evaluated. Julien Arzi provided numerical evaluations of several estimators for comparison to the proposed estimator as a function of SNR with uniformly-distributed frequencies [6]. In this paper we examine the inter-bin performance of the proposed estimator and compare it to the previously mentioned estimators.

Estimator Descriptions

The estimators generally estimate δ, the fractional interbin correction to the discrete frequency of the DFT maximizer at index k_p, so that f = $(k_p+\delta)/N$ is the normalized frequency. The DFT coefficients of the tone are designed R_k. The estimators are as follows, with $k = k_p$:

Jacobsen [1]: Equation (1)

$$\widehat{\delta_J} = Re\left\{\frac{(R_{k-1} - R_{k+1})}{(2R_k - R_{k-1} - R_{k+1})}\right\}$$

Candan's first [2]: Equation (2)

$$\hat{\delta}_{C1} = \frac{\tan\,(\pi/N)}{\pi/N}\widehat{\delta_J}$$

Candan's second [3]: Equation (3)

$$\hat{\delta}_{C2} = \frac{\mathrm{atan}\,(\hat{\delta}_{C1}\pi/N)}{\pi/N}$$

Liao [4]: Equation (4)

$$D_{qmj} = \left(\frac{3\pi^4}{N^4}\hat{\delta}_{qmj} + \sqrt{\frac{4\pi^6}{N^6}\left(1-\frac{\pi^2}{3N^2}\right)^3 + \frac{9\pi^8}{N^8}\hat{\delta}_{qmj}{}^2}\right)^{1/3}$$

where $\hat{\delta}_{qmj}$ is any of the Quinn, MacLeod or Jacobsen estimators, for example, $\hat{\delta}_J$.

Then, for the QMJ case:

Equation (5)

$$\hat{\delta}_{rqmj} = \frac{D_{qmj}}{2^{1/3}\frac{\pi^2}{N^2}} - \frac{2^{1/3}\left(1-\frac{\pi^2}{3N^2}\right)}{D_{qmj}}$$

and for the Candan (2) case:

Equation (6)

$$\hat{\delta}_{rC1} = \frac{D_{c1}}{2^{1/3}\frac{\pi^2}{N^2}} - \frac{2^{1/3}}{D_{c1}}$$

where

$$D_{C1} = \left(\frac{3\pi^4}{N^4}\hat{\delta}_{C1} + \sqrt{\frac{4\pi^6}{N^6} + \frac{9\pi^8}{N^8}\hat{\delta}_{C1}{}^2}\right)^{1/3}$$

Cedron [5]:

Equation (7)

$$R1 = e^{-j2\pi/N}$$

Equation (8)

$$B = -\cos\big((k-1)2\pi/N\big)R_{k-1} + (1 - R1)\cos(k2\pi/N)R_k - R1\cos\big((k+1)2\pi/N\big)R_{k+1}$$

Equation (9)

$$C = -R_{k-1} + (1 + R1)R_k - R1R_{k+1}$$

Equation (10)

$$\hat{f}_{Ce} = \cos^{-1}\left(\mathrm{Re}\left(\frac{\mathrm{B}}{\mathrm{C}}\right)\right)\mathrm{N}/2\pi$$

Because Cedron's estimator uses the indices, k, k-1, and k+1 directly in the calculation it estimates normalized frequency, f, not the bin offset, δ.

Evaluation

The estimators were compared using evaluation routines written in Matlab/Octave. Ten frequencies are tested, sweeping a complex-valued tone in 1/10th bin increments from bin 9 to bin 10 of a 64-pt DFT, with 100 trials with randomized phase in the no-noise cases. In all cases the vertical axis is in bins, and the horizontal axis is test number with test 1 having the frequency centered on bin 9 incrementing in 1/10th bin steps and the 10th at bin 9.9. The left-hand plot is estimator variance and the right-hand plot is estimator bias. The estimators are identified as follows:

Jacobsen (1) - Black circles

Candan (2) - Blue x's

Candan (3) - Magenta dots

Liao applied to Jacobsen (5) - Red circles

Liao applied to Candan (first) (6) - Green line

Cedron (10) - Black +

First the no-noise case.

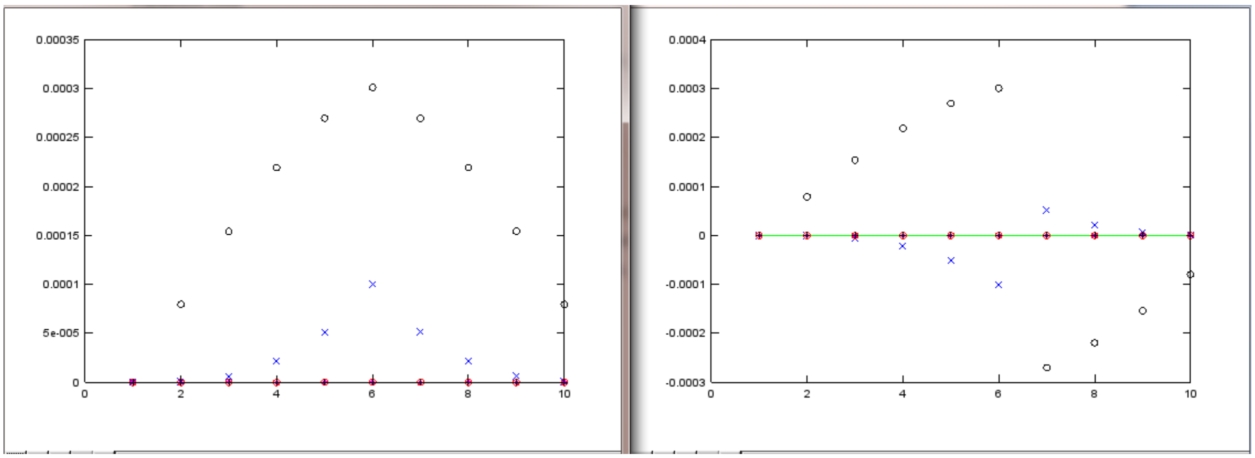

Figure 1. Estimator variance (left) and bias (right) for the no-noise cases. As expected, Jacobsen's estimator has the highest variance and bias in these conditions, with Candan (2) next. The error variance in these cases is essentially due to the bias. The additional corrections and complexity in the other estimators reduce the variance and bias.

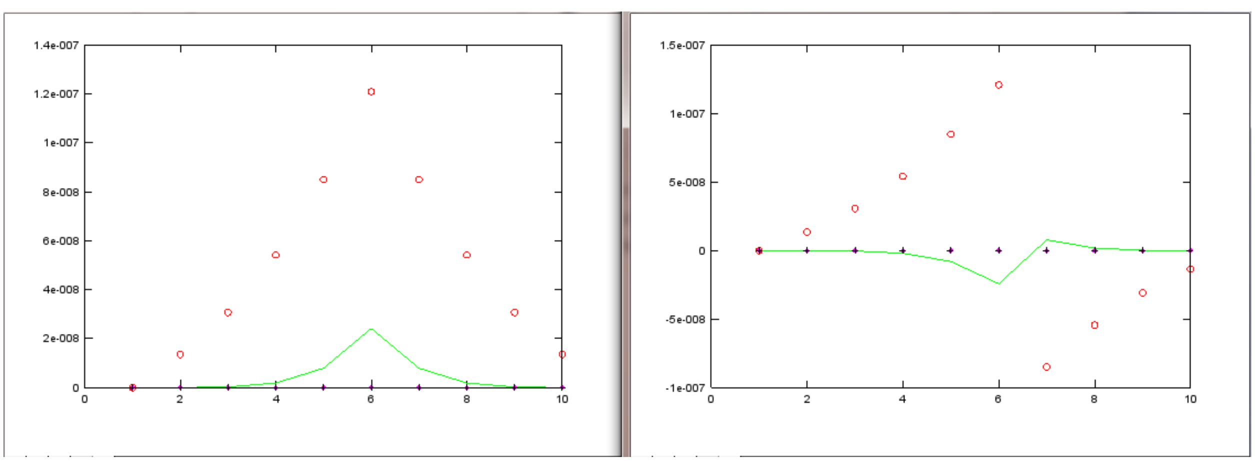

Figure 2. This is Figure 1 with Jacobsen's and Candan's first estimators removed to improve the vertical scale resolution. The red circles are Jacobsen's estimator with Liao's correction, and the green line is Candan's first estimator with Liao's correction. Candan's second estimator and Cedron's error performance metrics are much smaller.

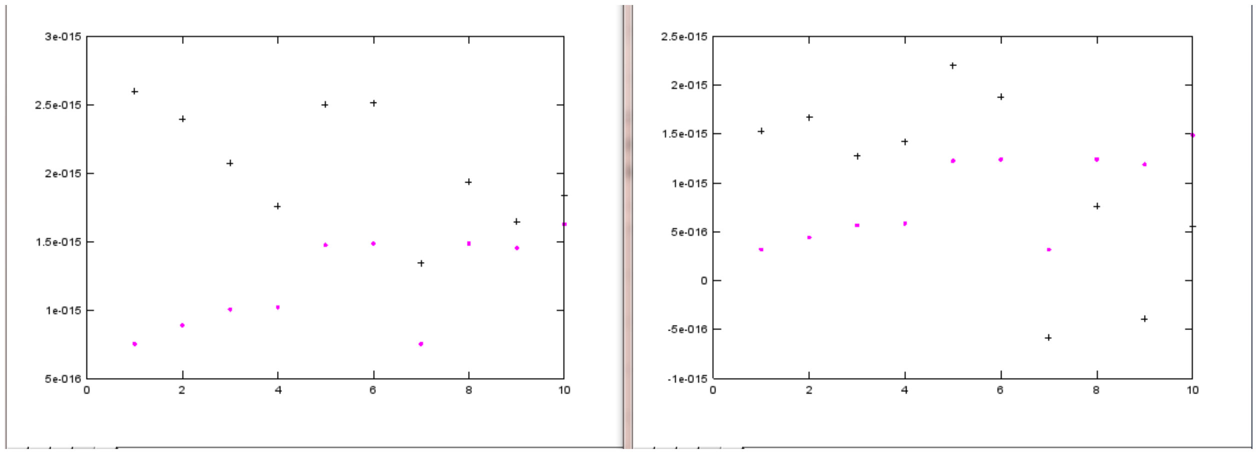

Figure 3.   This is Figure 2 with Liao's corrected estimators removed.   Only Candan's second and Cedron's estimators remain and are nearly indistinguishable.   The apparent noise is likely due to the numeric precision of the simulations.

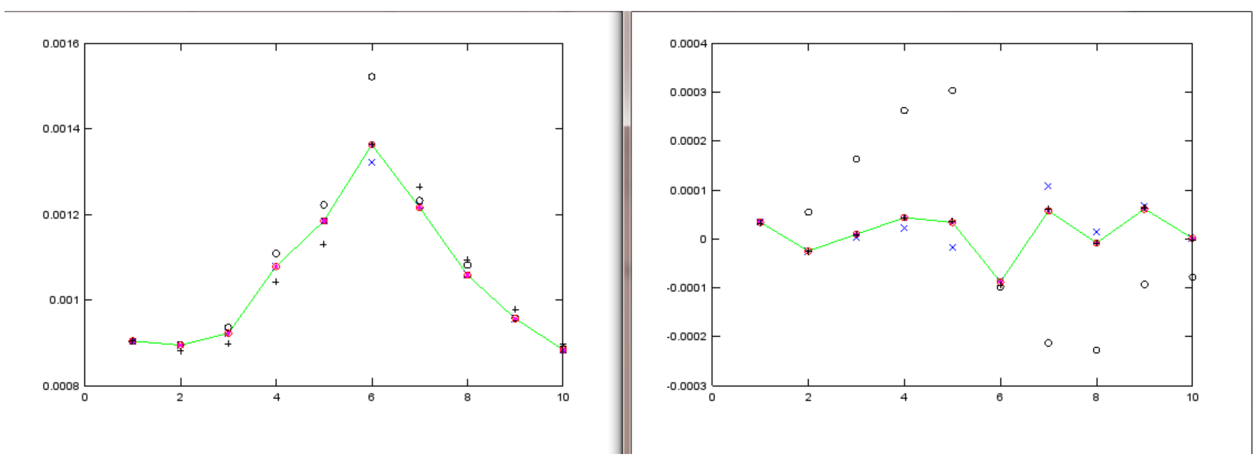

Figure 4.   The variance (left) and bias (right) of all of the compared estimators at SNR = 37dB. The number of trials per frequency is 1000.

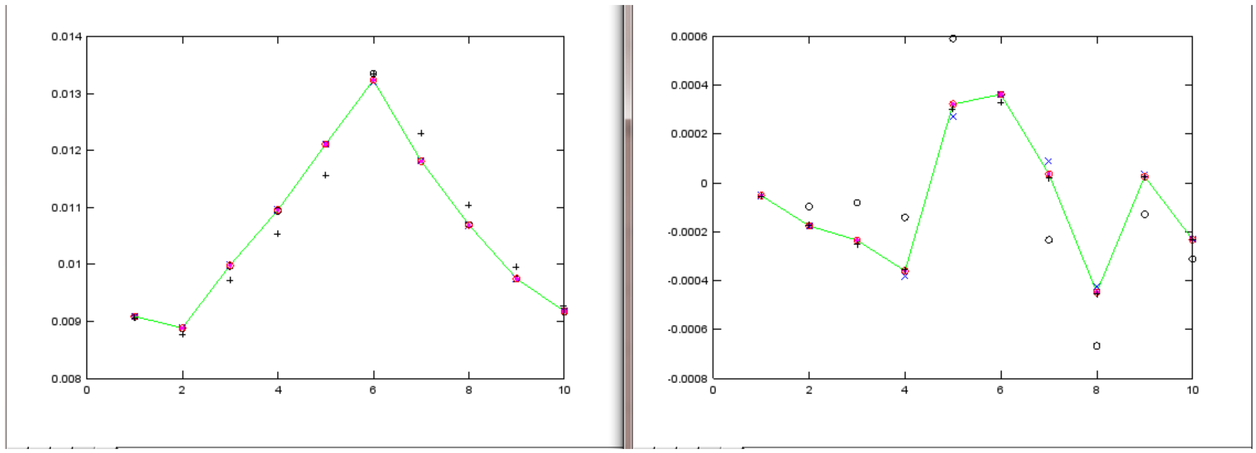

Figure 5. The variance (left) and bias (right) of all of the compared estimators at SNR = 17dB. The number of trials per frequency is 1000.

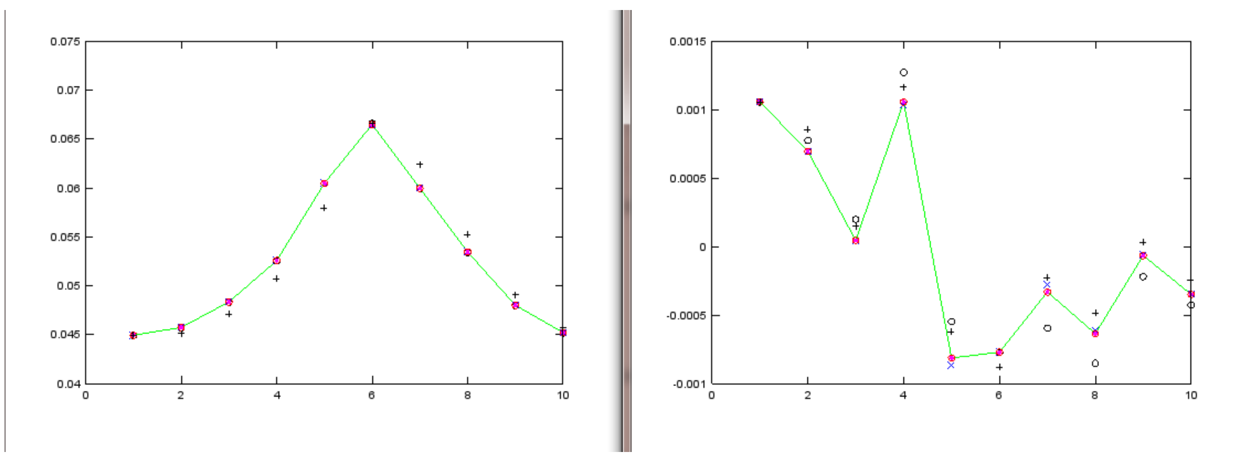

Figure 6. The variance (left) and bias (right) of all of the compared estimators at SNR = 3dB. The number of trials per frequency is 5000.

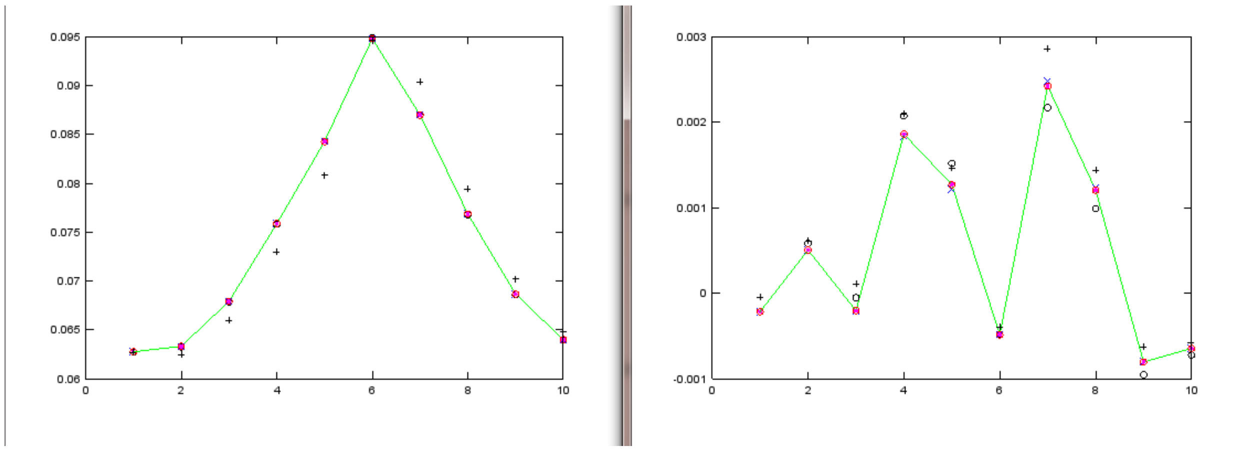

Figure 7. The variance (left) and bias (right) of all of the compared estimators at SNR = 0dB. The number of trials per frequency is 5000.

Conclusion

The various estimators are compared and show similar performance at low SNR. As SNR increases and the bias dominates the error performance of the simpler estimators, the more complex estimators maintain low bias. In the no-noise case Candan's second estimator and Cedron's maintain much lower bias than the other estimators. While Liao's corrections provide significant reduction in estimator bias for both Jacobsen and Candan's first estimator, it does so at the cost of much higher complexity than Candan's second estimator. Although Candan's second estimator and Cedron's both have similar, very low, bias in the no-noise case, the complexity of Candan's estimator is substantially lower than Cedron's by a large margin. At low SNR all estimators are essentially indistinguishable in performance.

References:

[1] Jacobsen, E., Kootsookos, P., "Fast, Accurate Frequency Estimators," IEEE Signal Processing Magazine, Vol. 24, Issue 3, pp. 123-125, May, 2007

[2] C. Candan, "A Method For Fine Resolution Frequency Estimation From Three DFT Samples," *IEEE Signal Processing Letters*, Vol. 18, No.6, p. 351-354, June 2011.

[3] C. Candan, "Analysis and Further Improvement of Fine Resolution Frequency Estimation Method From Three DFT Samples," *IEEE Signal Processing Letters*, vol.20, no.9, p. 913–916, Sept. 2013.

[4] Jan-Ray Liao, Shyng Lo, “Analytical solutions for frequency estimators by interpolation of DFT coefficients,” Signal Processing, vol.100, pp.93-100, July 2014.

[5] Cedron, "Exact Frequency Formula For a Pure Real Tone in DFT", http://www.dsprelated.com/showarticle/773.php

[6] Julien Arzi on-line article, "Comparison of different frequency estimation algorithms", http://www.tsdconseil.fr/log/scriptscilab/festim/freqestim-comp3.pdf